Algebraic models of simple type theory

(A polynomial approach)

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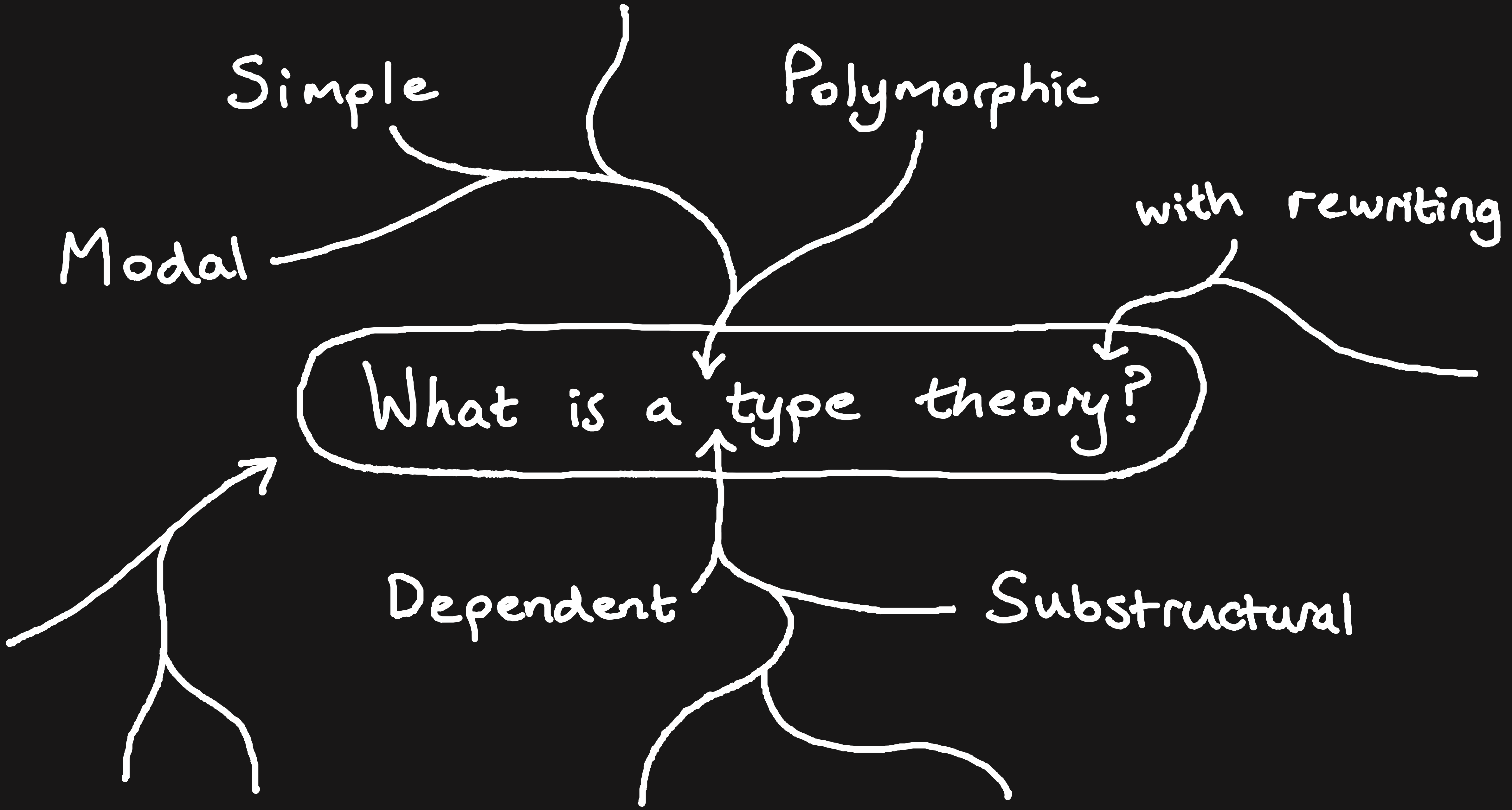
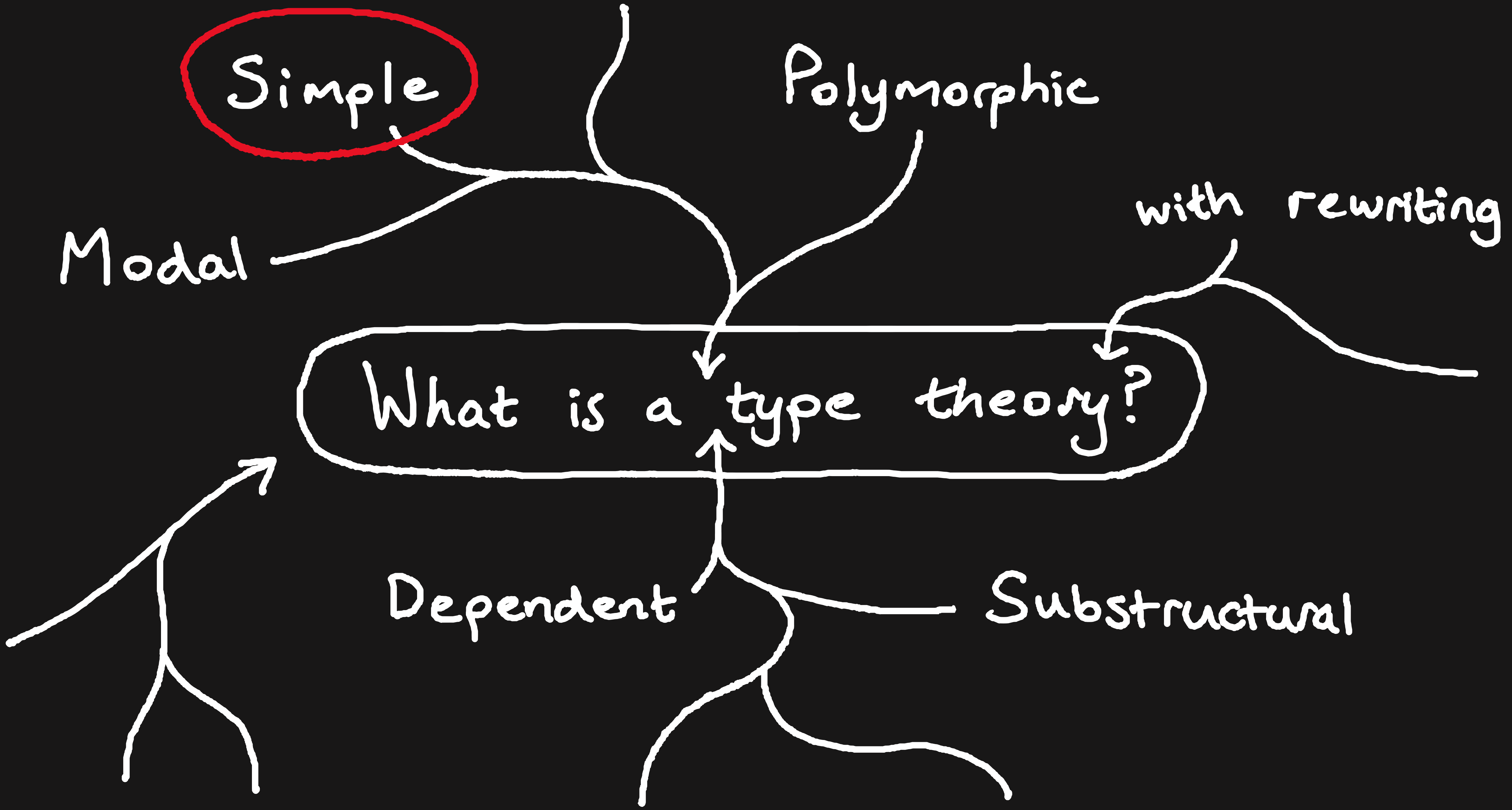
What is a type theory?


What is a type theory?
Simple
Polymorphic
with rewriting
Modal
Dependent
Substructural


Simple
Polymorphic
Modal
with rewriting
What is a type theory?
Dependent
Substructural

What is a <u>simple</u> type theory?

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : A \times B}\ (\times\text{-INTRO})$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : A \Rightarrow B}\ (\Rightarrow\text{-INTRO})$$

What is a <u>simple</u> type theory?

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : A \times B} \;(\times\text{-INTRO})$$

algebraic structure on types

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : A \Rightarrow B} \;(\Rightarrow\text{-INTRO})$$

What is a <u>simple</u> type theory?

algebraic structure on terms

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algebraic structure on types

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algebraic structure on types

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.\,t) : A \Rightarrow B}\ (\Rightarrow\text{-INTRO})$$

binding algebraic structure on terms

What is a <u>simple</u> type theory?

$$\frac{\Gamma, x:A \vdash t:B \qquad \Gamma \vdash a:A}{\Gamma \vdash \mathrm{app}(\mathrm{abs}(x.t), a) \equiv t[^a/_x] : B}\ (\Rightarrow\text{-}\beta)$$

What is a simple type theory?

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathrm{app}(\mathrm{abs}(x.t), a) \equiv t[{}^{a}/_{x}] : B}\,(\Rightarrow\text{-}\beta)$$

equations on types & terms

What is a simple type theory?

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathrm{app}(\mathrm{abs}(x.t), a) \equiv t[a/x] : B}\ (\Rightarrow\text{-}\beta)$$

equations on types & terms

capture-avoiding substitution

What is a <u>simple</u> type theory?

- Algebraic type operators.
- Binding algebraic term operators.

- Substitution on terms.
- Equations on types.
- Equations on terms.

What is a simple type theory?

- Type operators.
- Binding term operators.

- Substitution on terms.
- Equations on types.
- Equations on terms.

E.g. Unityped λ-calculus, STLC, computational LC, predicate logic, functional arrows, partial differentiation, STLC with sums, universal algebras, etc.

What is a <u>simple</u> type theory?

- Type operators.
- Binding term operators.

Simply-typed syntax

- Substitution on terms.
- Equations on types.
- Equations on terms.

Simple type theory

Signatures & equational presentations

- Presentations for types as in universal algebra.
- Presentations for terms includes binding & substitution.

induce polynomial functors Σ_{ty} & Σ_{tm}

(Details omitted.)

Algebraic models of simple type theories

$$\Gamma, x : A \vdash t : B$$

Algebraic models of simple type theories

context structure

type structure

$$\Gamma, x : A \vdash t : B$$

term structure

Algebraic models of simple type theories

context structure

type structure

$$\Gamma, x : A \vdash t : B$$

term structure

(inc. substitution structure)

$$t[{}^{a}/_{x}]$$

Type algebras

$$
\begin{tikzcd}
\Sigma_{ty}(S) \arrow[d, "[\![ty]\!]"] \\
S
\end{tikzcd}
$$

(as in universal algebra)

Type algebras

$$\begin{tikzcd} \Sigma_{ty}(S) \arrow[d, "[\![ty]\!]"] \\ S \end{tikzcd}$$

set of sorts $\longrightarrow S$

(as in universal algebra)

Type algebras

(polynomial) functor induced by type signature

$$\begin{array}{c} \Sigma_{ty}(S) \\ \big\downarrow {\scriptstyle [\![ty]\!]} \\ S \end{array}$$

set of sorts $\longrightarrow$ S

(as in universal algebra)

Type algebras

(polynomial)
functor induced
by type signature

$$
\begin{tikzcd}
\Sigma_{ty}(S) \arrow[d, "[\![ty]\!]"] \\
S
\end{tikzcd}
$$

denotation of
type operators

set of sorts $\longrightarrow S$

(as in universal algebra)

Type algebras

E.g. STLC.

$$\begin{tikzcd} S^2 + S^2 + 1 \arrow[d, "{[\![\mathrm{Prod}, \mathrm{Fun}, \mathrm{Unit}]\!]}"] \\ S \end{tikzcd}$$

Context structures

$$S \xhookrightarrow{\langle - \rangle} \mathbb{C}$$

$$1 \in \mathbb{C}$$

$$\Gamma \times \langle A \rangle \in \mathbb{C} \qquad (\Gamma \in \mathbb{C},\ A \in S)$$

Context structures

category of contexts

$$S \xhookrightarrow{\langle - \rangle} \mathbb{C}$$

$$1 \in \mathbb{C}$$

$$\Gamma \times \langle A \rangle \in \mathbb{C} \qquad (\Gamma \in \mathbb{C},\ A \in S)$$

Context structures

Set of sorts

sorts as single-variable contexts

category of contexts

$$S \xhookrightarrow{\langle - \rangle} \mathbb{C}$$

E.g. $\langle A \rangle = (x : A) \quad (A \in S)$

$$1 \in \mathbb{C}$$

$$\Gamma \times \langle A \rangle \in \mathbb{C} \qquad (\Gamma \in \mathbb{C},\ A \in S)$$

Context structures

set of sorts

sorts as single-variable contexts

category of contexts

$$S \xhookrightarrow{\langle - \rangle} \mathbb{C}$$

empty context

$$1 \in \mathbb{C}$$

$$\Gamma \times \langle A \rangle \in \mathbb{C} \qquad (\Gamma \in \mathbb{C},\ A \in S)$$

Context structures

set of sorts

sorts as single-variable contexts

category of contexts

$$S \xrightarrow{\langle - \rangle} \mathbb{C}$$

empty context

$$1 \in \mathbb{C}$$

context extension

$$\Gamma \times \langle A \rangle \in \mathbb{C} \qquad (\Gamma \in \mathbb{C},\ A \in S)$$

Typed term structures

$$\begin{tikzcd} T \arrow[d, "\tau"] \\ S \end{tikzcd}$$

Typed term structures

$$\begin{tikzcd} T \arrow[d, "\tau"] \\ S \end{tikzcd}$$

set of sorts $\longrightarrow S$

Typed term structures

$$\begin{tikzcd} T \arrow[d, "\tau"] \\ S \end{tikzcd}$$

set of terms

set of sorts

Typed term structures

set of terms

$$\begin{tikzcd} T \arrow[d, "\tau"] \\ S \end{tikzcd}$$

type for each term

set of sorts

Typed term structures

presheaf of terms

$$\begin{tikzcd} T \arrow[d, "\tau"] \\ S \end{tikzcd}$$

type for each term in context

set of sorts

(in $\widehat{\mathbb{C}}$)

Typed term structures

Set of terms
in context Γ
(natural in Γ)

$$\begin{tikzcd} T(\Gamma) \arrow[d, "\tau_\Gamma"] \\ S(\Gamma) = S \end{tikzcd} \qquad (\Gamma \in \mathbb{C})$$

Term algebras (in $\widehat{\mathbb{C}}/S$)

$$\begin{tikzcd}
\Sigma_{tm}(T) \arrow[r, "[\![tm]\!]"] \arrow[dr] & T \arrow[d, "\tau"] \\
& S
\end{tikzcd}$$

Term algebras (in $\widehat{\mathbb{C}}/S$)

$$
\begin{tikzcd}
\Sigma_{tm}(T) \arrow[r, "{[\![tm]\!]}"] \arrow[dr] & T \arrow[d, "\tau"] \\
 & S
\end{tikzcd}
$$

(polynomial) functor induced by term signature

Term algebras (in $\widehat{\mathbb{C}}/S$)

denotation of term operators

$$\begin{tikzcd} \Sigma_{tm}(T) \arrow[r, "[\![tm]\!]"] \arrow[dr] & T \arrow[d, "\tau"] \\ & S \end{tikzcd}$$

(polynomial) functor induced by term signature

Term algebras (in $\widehat{\mathbb{C}}/S$)

E.g. product introduction.

$$\begin{tikzcd}
T^2 \arrow[r, "{[\![\mathrm{pair}]\!]}"] \arrow[d, "\tau^2"'] & T \arrow[d, "\tau"] \\
S^2 \arrow[r, "{[\![\mathrm{Prod}]\!]}"'] & S
\end{tikzcd}$$

Term algebras (in $\widehat{\mathbb{C}}/S$)

E.g. product introduction.

$$\Sigma_{tm}(T) = T^2 \xrightarrow{[\![\mathrm{pair}]\!]} T$$

$$\begin{tikzcd}
\Sigma_{tm}(T) = T^2 \arrow[r, "{[\![\mathrm{pair}]\!]}"] \arrow[d, "\tau^2"'] \arrow[dr] & T \arrow[d, "\tau"] \\
S^2 \arrow[r, "{[\![\mathrm{Prod}]\!]}"'] & S
\end{tikzcd}$$

Substitution structure (part i)

$$\begin{tikzcd} V \arrow[r, "var"] \arrow[dr, "\nu"'] & T \arrow[d, "\tau"] \\ & S \end{tikzcd}$$

Substitution structure (part i)

variables in context Γ

$$\begin{tikzcd} V(\Gamma) \arrow[r, "var_\Gamma"] \arrow[dr, "\nu_\Gamma"'] & T(\Gamma) \arrow[d, "\tau_\Gamma"] \\ & S \end{tikzcd}$$

Substitution structure (part i)

variables in context Γ

$$\begin{tikzcd} V(\Gamma) \arrow[r, "\mathrm{var}_\Gamma"] \arrow[dr, "\nu_\Gamma"'] & T(\Gamma) \arrow[d, "\tau_\Gamma"] \\ & S \end{tikzcd}$$

$V(\Gamma) = \coprod_{A \in S} \mathbb{C}(\Gamma, \langle A \rangle)$

$\nu_\Gamma = \pi$

Substitution structure (part i)

variables in context Γ

variable projection

$$\begin{tikzcd} V(\Gamma) \arrow[r, "\mathrm{var}_\Gamma"] \arrow[dr, "\nu_\Gamma"'] & T(\Gamma) \arrow[d, "\tau_\Gamma"] \\ & S \end{tikzcd}$$

Substitution structure (part ii)

$$\begin{tikzcd}
\coprod_{A,B \in S} T_B^{V_A} \times T_A \arrow[r, "\mathrm{subst}"] \arrow[dr] & T \arrow[d, "\tau"] \\
& S
\end{tikzcd}$$

Substitution structure (part ii)

$$\begin{tikzcd} \coprod\limits_{A,B \in S} T_B^{V_A} \times T_A \arrow[r, "\mathrm{subst}"] \arrow[dr] & T \arrow[d, "\tau"] \\ & S \end{tikzcd}$$

terms of type A

$$\begin{tikzcd} T_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\tau"] \\ 1 \arrow[r, "A"'] & S \end{tikzcd}$$

Substitution structure (part ii)

$$\begin{tikzcd} \coprod_{A,B\in S} T_{B\times}^{V_A} T_A \arrow[r, "\mathrm{subst}"] \arrow[dr] & T \arrow[d, "\tau"] \\ & S \end{tikzcd}$$

$$T_B^{V_A}(\Gamma) \cong T_B(\Gamma, x : A)$$

Substitution structure (part ii)

$$\begin{tikzcd}
\coprod_{A,B\in S} T_B^{V_A}\times T_A \arrow[r, "\mathrm{subst}"] \arrow[dr] & T \arrow[d, "\tau"] \\
& S
\end{tikzcd}$$

$$\frac{\Gamma, x:A \vdash t:B \qquad \Gamma \vdash a:A}{\Gamma \vdash t[a/x]:B}$$

Substitution structure (part ii)

$$\begin{tikzcd}
\coprod_{A,B \in S} T_B^{V_A} \times T_A \arrow[r, "\mathrm{subst}"] \arrow[dr] & T \arrow[d, "\tau"] \\
& S
\end{tikzcd}$$

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash a : A}{\Gamma \vdash t[a/x] : B}$$

(+ axioms & coherence laws)

Algebraic models of simple type theories

- Type algebra.
- Context structure.
- Term algebra.

Simply-typed syntax

- Substitution structure.
- ... subject to equations.

Simple type theory

Meta-theory

Meta-theory

- Initiality theorem.
  (Syntactic model is initial.)

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  (Syntactic model is initial.)
- Substitution lemma.
  (Substitution is admissible.)

Meta-theory

- Initiality theorem.
  (Syntactic model is initial.)
- Substitution lemma.
  (Substitution is admissible.)
- General Lambek theorem.
  (Multisubstitutional models of simple type theories are equivalent to cartesian multicategories with corresponding structure.)

Summary

- A framework for simple type theories.
- Syntax (signatures & presentations) & semantics (models for simply typed syntax & simple type theories).
- Meta-theory (initiality theorem, substitution lemma, general Lambek theorem).

Summary

- A framework for simple type theories.
- Syntax (signatures & presentations) & semantics (models for simply typed syntax & simple type theories).
- Meta-theory (initiality theorem, substitution lemma, general Lambek theorem).
- A new perspective on natural deduction rules inducing polynomials.

Lightning introduction to polynomials

(1) $$I \xleftarrow{s} A \xrightarrow{f} B \xrightarrow{t} J \qquad (\text{in } \mathcal{C})$$

polynomial

Lightning introduction to polynomials

① $$I \xleftarrow{s} A \xrightarrow{f} B \xrightarrow{t} J \quad (\text{in } \mathcal{C})$$

polynomial

② $$(X_i \mid i \in I) \mapsto \left(\sum_{b \in B_j} \prod_{a \in A_b} X_{s(b)} \,\middle|\, j \in J \right)$$

$$\mathcal{C}/I \longrightarrow \mathcal{C}/J$$

polynomial functor

Lightning introduction to polynomials

① $I \xleftarrow{s} A \xrightarrow{f} B \xrightarrow{t} J$ (in $\mathcal{C}$)

polynomial

sum of products

with reindexing

② $(X_i \mid i \in I) \mapsto \left(\sum_{b \in B_j} \prod_{a \in A_b} X_{s(b)} \mid j \in J \right)$

$\mathcal{C}/I \longrightarrow \mathcal{C}/J$

polynomial functor

Natural deduction rules induce polynomials

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : \mathrm{Prod}(A, B)}\ (\text{Prod-INTRO})$$

Natural deduction rules induce polynomials

$$(\forall A, B)\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : \mathrm{Prod}(A, B)}\ (\text{Prod-INTRO})$$

Natural deduction rules induce polynomials

$$(\forall \underline{A, B})\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : \mathrm{Prod}(A, B)}\ (\text{Prod-INTRO})$$

type metavariables

$$\begin{array}{ccc} A & & B \\ \downarrow & & \downarrow \\ S & \times & S \end{array}$$

Natural deduction rules induce polynomials

$$(\forall A, B)\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : \underline{\mathrm{Prod}(A, B)}}\ (\text{Prod-INTRO})$$

conclusion type

$$S \times S \xrightarrow{[\![\mathrm{Prod}]\!]} S$$

$$(A, B) \longmapsto [\![\mathrm{Prod}]\!](A, B)$$

Natural deduction rules induce polynomials

two premisses

$$(\forall A, B)\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : \mathrm{Prod}(A, B)}\ (\text{Prod-INTRO})$$

$$\overbrace{S \times S}^{\Gamma \vdash a : A} + \overbrace{S \times S}^{\Gamma \vdash b : B} \longrightarrow S \times S \xrightarrow{[\![\mathrm{Prod}]\!]} S$$

Natural deduction rules induce polynomials

$$(\forall A, B)\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : \mathrm{Prod}(A, B)}\ (\text{Prod-INTRO})$$

unify $A_1 = A_2$
and $B_1 = B_2$

$$\underset{\overset{\uparrow}{A_1}}{\overset{A_1}{\downarrow}}$$

$$\overset{\overset{A_1}{\downarrow}}{S} \times \overset{\overset{B_1}{\downarrow}}{S} + \underset{\underset{A_2}{\uparrow}}{S} \times \underset{\underset{B_2}{\uparrow}}{S} \xrightarrow{\nabla_2} S \times S \xrightarrow{[\![\mathrm{Prod}]\!]} S$$

Natural deduction rules induce polynomials

$$(\forall A, B)\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : \mathrm{Prod}(A, B)}\ (\text{Prod-INTRO})$$

$$S \xleftarrow{[\pi_1, \pi_2]} S \times S + S \times S \xrightarrow{\nabla_2} S \times S \xrightarrow{[\![\mathrm{Prod}]\!]} S$$

A

B

Natural deduction rules induce polynomials

$$(\forall A, B)\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pair}(a, b) : \mathrm{Prod}(A, B)}\ (\text{Prod-INTRO})$$

$$S \xleftarrow{[\pi_1, \pi_2]} S \times S + S \times S \xrightarrow{\nabla_2} S \times S \xrightarrow{[\![\mathrm{Prod}]\!]} S$$

Natural deduction rules induce polynomials

An algebra for the polynomial functor induced by $S \xleftarrow{[\pi_1,\pi_2]} S^2 + S^2 \xrightarrow{\nabla_2} S^2 \xrightarrow{[\![\mathrm{Prod}]\!]} S$ is given by:

$$
\begin{tikzcd}
 & & 2 & \\
\coprod_{A,B\in S} T_A \times T_B \arrow[rr, "{[\![\mathrm{pair}]\!]}"] \arrow[d, "\pi"'] & & T \arrow[d, "\tau"] & 1 \\
S^2 \arrow[rr, "{[\![\mathrm{Prod}]\!]}"'] & & S &
\end{tikzcd}
$$

Natural deduction rules induce polynomials

An algebra for the polynomial functor induced by $S \xleftarrow{[\pi_1, \pi_2]} S^2 + S^2 \xrightarrow{\nabla_2} S^2 \xrightarrow{[\![\mathrm{Prod}]\!]} S$ is given by:

$$
\begin{tikzcd}
T^2 \arrow[r, "{[\![\mathrm{pair}]\!]}\ 2"] \arrow[d, "\tau^2"'] & T \arrow[d, "\tau"] \arrow[dr, phantom, "1"] & \\
S^2 \arrow[r, "{[\![\mathrm{Prod}]\!]}"'] & S &
\end{tikzcd}
$$

Natural deduction rules induce polynomials

$$\frac{\Gamma,\ x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \mathrm{Fun}(A,B)}\ (\text{Fun-INTRO})$$

Natural deduction rules induce polynomials

$$(\forall A, B)\ \frac{\Gamma,\ x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \mathrm{Fun}(A,B)}\ (\text{Fun-INTRO})$$

Natural deduction rules induce polynomials

$$(\forall \underbrace{A, B}_{\text{type metavariables}}) \quad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \mathrm{Fun}(A, B)} \ (\text{Fun-INTRO})$$

$$\underset{A}{S} \times \underset{B}{S}$$

Natural deduction rules induce polynomials

$$(\forall A, B)\ \frac{\Gamma,\ x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \underline{\mathrm{Fun}(A,B)}}\ (\text{Fun-INTRO})$$

conclusion type

$[\![\mathrm{Fun}]\!]$

$$S \times S \longrightarrow S$$

$$(A, B) \longmapsto [\![\mathrm{Fun}]\!](A,B)$$

Natural deduction rules induce polynomials

one premiss

$$(\forall A, B)\ \frac{\Gamma,\ x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \mathrm{Fun}(A,B)}\ (\text{Fun-INTRO})$$

$$\overbrace{V \times S}^{\Gamma,\, x:A \vdash t:B}$$

$$S \times S \xrightarrow{[\![\mathrm{Fun}]\!]} S$$

Natural deduction rules induce polynomials

bound variable

$$(\forall A, B)\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \mathrm{Fun}(A, B)}\ (\text{Fun-INTRO})$$

the type metavariable A is bound in this premiss

$$V \times S \qquad\qquad S \times S \xrightarrow{[\![\mathrm{Fun}]\!]} S$$

Natural deduction rules induce polynomials

$$(\forall A, B)\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \mathrm{Fun}(A,B)}\ (\text{Fun-INTRO})$$

forget binding structure

$$V \times S \xrightarrow{\nu \times id} S \times S \xrightarrow{[\![\mathrm{Fun}]\!]} S$$

Natural deduction rules induce polynomials

$$(\forall A, B) \quad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \mathrm{Fun}(A, B)} \; (\text{Fun - INTRO})$$

$$\begin{tikzcd} S & V \times S \arrow[l, "\pi_2"'] \arrow[l, bend right, "B"'] \arrow[r, "\nu \times \mathrm{id}"] & S \times S \arrow[r, "[\![\mathrm{Fun}]\!]"] & S \end{tikzcd}$$

Natural deduction rules induce polynomials

$$(\forall A, B) \ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \mathrm{abs}(x.t) : \mathrm{Fun}(A,B)} \ (\text{Fun-INTRO})$$

$$S \xleftarrow{\pi_2} V \times S \xrightarrow{\nu \times id} S \times S \xrightarrow{[\![\mathrm{Fun}]\!]} S$$

Natural deduction rules induce polynomials

An algebra for the polynomial functor induced by $S \xleftarrow{\pi_2} V \times S \xrightarrow{\nu \times id} S^2 \xrightarrow{[\![Fun]\!]} S$ is given by:

$$
\begin{tikzcd}
 & & 2 & \\
\coprod_{A,B \in S} T_B^{V_A} \arrow[rr, "{[\![abs]\!]}"] \arrow[d, "\pi"'] & & T \arrow[d, "\tau"] & 1 \\
S^2 \arrow[rr, "{[\![Fun]\!]}"'] & & S &
\end{tikzcd}
$$

Natural deduction rules induce polynomials

An algebra for the polynomial functor induced by $S \xleftarrow{\pi_2} V \times S \xrightarrow{v \times id} S^2 \xrightarrow{[\![Fun]\!]} S$ is given by:

$$\begin{tikzcd}
\coprod_{A,B \in S} T_B^{V_A} \arrow[r, "{[\![abs]\!]}"] \arrow[d, "\pi"'] & T \arrow[d, "\tau"] \\
S^2 \arrow[r, "{[\![Fun]\!]}"'] & S
\end{tikzcd}$$

$T_B(\Gamma, x : A)$

Summary

- A framework for simple type theories.
- Syntax (signatures & presentations) & semantics (models for simply typed syntax & simple type theories).
- Meta-theory (initiality theorem, substitution lemma, general Lambek theorem).
- A new perspective on natural deduction rules inducing polynomials, whose algebras are models of the rule.